

Integral structures on the finite part $H_f^1(K, V)$ of a crystalline representation

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Abstract

We study integral structures of crystalline representations over an unramified extension K/\mathbb{Q}_p with the help of an auxiliary ring \mathbf{A}_{exp} . This ring has the nice property that it contains the the fundamental period (and its inverse) of p -adic Hodge theory, up to powers of p . We establish an exact sequence using \mathbf{A}_{exp} and Frobenii on its filtration, give a link to Fontaine-Laffaille modules and the Bloch-Kato fundamental exact sequence and finally compute the integral finite part of a lattice of a crystalline representation, giving a connection to the local L -function of V .

1 Introduction

In their fundamental work [1], Bloch and Kato used and developed many techniques of what is now usually called p -adic Hodge theory, initiated before in large parts by Fontaine. Bloch and Kato's focus was the development of a general conjecture concerning special values of L -functions, which culminated in their formulation of a version of the Tamagawa number conjecture.

Working locally at a fixed prime p and a fixed finite extensions K/\mathbb{Q}_p with absolute Galois group G_K , we take a closer look at the computations done in sections 3 and 4 in loc.cit., which depend in certain situations on the property that the p -adic representation V of G_K under consideration is “in the Fontaine-Laffaille range”. This is a condition on the filtration of the filtered φ -module associated to V .

We introduce an auxiliary integral ring \mathbf{A}_{exp} , which, after inverting p , computes the module $\mathbf{D}_{\text{cris}}(V)$ of a crystalline representation, if one fixes a G_K -equivariant lattice $T \subset V$. A nice property of this integral version is that it contains already (up to some p -powers) the inverses of the fundamental period t , so that no awkward twisting to a positive representation is necessary. Note that simply inverting t in for example \mathbf{A}_{cris} implies that p is then also already inverted, which leaves the integral world.

Using this ring, we show that one can construct a finite rank Fontaine-Laffaille module $\mathbf{D}_{\text{exp}}(T)$ out of T , which is used to connect the p -adic valuation of the special value at $s = 0$ of the local L -function $P(V, p^{-s})$ to a certain measure on this Fontaine-Laffaille module (via Bloch-Kato's fundamental exact sequence), without any condition on the filtration range of V :

Theorem 1.1. *Assume K/\mathbb{Q}_p is unramified and let V be a crystalline representation. Fix a G_K -equivariant lattice T in V and assume further that $P(V, 1) \neq 0$. Then:*

- a) $H^1(\mathbf{D}_{\exp}(T))[1/p] \cong H_e^1(K, V)$.
- b) $H^1(\mathbf{D}_{\exp}(T)) \cong H_e^1(K, T)$.
- c) $\exp_e : \mathbf{D}_{dR}(V)/\mathbf{D}_{dR}^0(V) \rightarrow H^1(K, V)$ coincides with the composite map

$$\begin{aligned} (\mathbf{D}_{\exp}(T)/\mathbf{D}_{\exp}^0(T))[1/p] &\xrightarrow{1-\varphi} (\mathbf{D}_{\exp}(T)/(1-\varphi^0)\mathbf{D}_{\exp}(T))[1/p] \\ &= H^1(\mathbf{D}_{\exp}(T))[1/p] \cong H^1(K, V), \end{aligned}$$

where the last canonical identification is explained in the proof.

Here, $H_e^1(K, -)$ denotes the exponential part of $H^1(K, -)$, that is, the image of the Bloch-Kato exponential map. As a corollary, we obtain:

Corollary 1.2. *Let μ be the Haar-measure on the finite-dimensional K -vector space $H^1(K, V)$ such that the image of the lattice*

$$\mathbf{D}_{\exp}(T)/\mathbf{D}_{\exp}^0(T) \subset \mathbf{D}_{dR}(V)/\mathbf{D}_{dR}^0(V) \xrightarrow{\sim} H_e^1(K, V) = H_f^1(K, V)$$

has measure 1. Then

$$\mu(H_e^1(K, T)) = |P(V, 1)|_p^{-1}.$$

2 Basic concepts from p -adic Hodge theory

Fix a prime number p . Let K/\mathbb{Q}_p be a finite extension of the p -adic numbers, and denote by G_K the absolute Galois group of K . Let K_0 be the maximal unramified subextension of K/\mathbb{Q}_p . Usually, V will denote a p -adic representation, that is, a finite dimensional \mathbb{Q}_p -vector space equipped with a continuous and linear G_K -action. Similarly, T will usually denote a G_K -stable \mathbb{Z}_p -lattice in V . Such lattices always exist. One is interested in the classification of such V and T , and Fontaine's rings have proven to be a powerful tool for this. We refer to [3] as a basic reference.

Let $\mathcal{O}_{\mathbb{C}_p}$ be the ring of integers of the completion of an algebraic closure \mathbb{C}_p of \mathbb{Q}_p . Let $\tilde{\mathbf{E}}^+ := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p$, which is a ring of characteristic p , equipped with a Frobenius $\varphi : x \mapsto x^p$, and a Galois action of $\sigma \in G_K$ via $(x_n) \mapsto (\sigma(x_n))$. If $x = (x_k) \in \tilde{\mathbf{E}}^+$, let $x^{(0)} = \lim_{m \rightarrow \infty} \hat{x}_m^{p^m}$, where $\hat{x}_m \in \mathcal{O}_{\mathbb{C}_p}$ is any lift of $x_m \in \mathcal{O}_{\mathbb{C}_p}/p$. This defines a non-archimedean valuation $v : x \mapsto v_p(x^{(0)})$ on $\tilde{\mathbf{E}}^+$.

Let $\tilde{\mathbf{A}}^+ = W(\tilde{\mathbf{E}}^+)$, the ring of Witt vectors of $\tilde{\mathbf{E}}^+$. This makes sense since $\tilde{\mathbf{E}}^+$ is perfect, since it is a perfection of the non-perfect ring $\mathcal{O}_{\mathbb{C}_p}/p$. $\tilde{\mathbf{A}}^+$ is a ring of characteristic 0 with Frobenius $\varphi : \sum_{n \geq 0} [x_n]p^n \mapsto \sum_{n \geq 0} [\varphi(x_n)]p^n$, and an action of G_K that is defined analogously.

One has the important ring homomorphism

$$\theta : \tilde{\mathbf{A}}^+ \rightarrow \mathcal{O}_{\mathbb{C}_p}, \quad \sum [x_n]p^n \mapsto \sum x_n^{(0)}p^n,$$

which arises conceptually in Fontaine's theory of universal thickenings.

We fix a system of p^n -th roots of unity $\varepsilon^{(n)} \in \mathbb{C}_p$ with $\varepsilon^{(0)} = 1, \varepsilon^{(1)} \neq 1$ and $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$. Then $\varepsilon = (\overline{\varepsilon^{(n)}}) \in \tilde{\mathbf{E}}^+$, where \bar{x} means reduction mod p . Let $\bar{\pi} := \varepsilon - 1$ (this notation is slightly unfortunate, but standard). One can show that $v(\bar{\pi}) = \frac{p}{p-1}$.

Let $\pi := [\varepsilon] - 1 \in \tilde{\mathbf{A}}^+$. Observe that $\pi \equiv \bar{\pi} \pmod{p}$, but $\pi \not\equiv [\bar{\pi}]$.

Further let $\tilde{p} \in \tilde{\mathbf{E}}^+$ with $\tilde{p}^{(0)} = -p$. Set $\xi = [\tilde{p}] + p \in \mathbf{A}^+$, i.e. $\theta(\xi) = -p + p = 0$. One can show that $\ker \theta = (\xi)$, since $\ker \theta \subset (\xi, p)$, using the fact that \mathbf{A}^+ is p -adically complete and that $\mathcal{O}_{\mathbb{C}_p}$ does not have any p -torsion. More generally, if $\xi' \in \tilde{\mathbf{A}}^+$ such that $\theta(\xi') = 0$ and $v(\bar{\xi}') = 1$, then $\ker \theta = (\xi')$.

One defines $\mathbf{B}_{\text{dR}}^+ = \varprojlim_n \tilde{\mathbf{A}}^+[1/p]/(\ker \theta)$. θ extends to \mathbf{B}_{dR}^+ , where $\ker \theta = (\xi)$ still holds. Let

$$t = \log(1 + \pi) = \sum_{n \geq 1} (-1)^{n+1} \frac{\pi^n}{n} \in \mathbf{B}_{\text{dR}}^+,$$

the fundamental period of p -adic Hodge theory. t only depends on the choice of a compatible system of p^n -th roots of unity. Interestingly, one has $\ker \theta = (t)$. This shows that \mathbf{B}_{dR}^+ is a complete discrete valuation ring with maximal ideal (t) . $\mathbf{B}_{\text{dR}} = \mathbf{B}_{\text{dR}}^+[1/t]$ possesses the filtration $\text{Fil}^i \mathbf{B}_{\text{dR}} = t^i \mathbf{B}_{\text{dR}}^+, i \in \mathbb{Z}$. Since $\mathbf{B}_{\text{dR}}^+/(t) \cong \mathbb{C}_p$, induced by the map θ , one has (algebraically, non-canonically) $\mathbf{B}_{\text{dR}} \cong \mathbb{C}_p[[t]]$. But observe that the topology on \mathbf{B}_{dR}^+ is defined via the inverse limit topology and the topology on $\tilde{\mathbf{A}}^+$, which is induced via the Witt-construction by the valuation topology on $\tilde{\mathbf{E}}^+$. With this topology, one still has a continuous action of G_K , but the action of φ does not extend to \mathbf{B}_{dR}^+ .

This being the case one considers the ring \mathbf{A}_{cris} , which is defined as the p -adic completion of the divided power envelope of $\tilde{\mathbf{A}}^+$ with respect to the ideal $\ker \theta$, i.e.

$$\tilde{\mathbf{A}}^+[\frac{a^m}{m!}; a \in \ker \theta]^\wedge = \tilde{\mathbf{A}}^+[\frac{\xi^m}{m!}] \subset \mathbf{B}_{\text{dR}}^+.$$

If $x \in \mathbf{A}_{\text{cris}}$, then we may write (non-uniquely) $x = \sum_n a_n \frac{\xi^n}{n!}$ with $a_n \in \tilde{\mathbf{A}}^+$ and $a_n \rightarrow 0$ p -adically. The map θ and the φ and G_K -action extend to \mathbf{A}_{cris} . Further, $t \in \mathbf{A}_{\text{cris}}$, since $\pi = b\xi$ ($\theta(\pi) = 0$) and

$$\frac{\pi^n}{n} = (n-1)! b^n \frac{\xi^n}{n!}, \quad (n-1)! \rightarrow 0.$$

Let $\mathbf{B}_{\text{cris}}^+ = \mathbf{A}_{\text{cris}}[1/p]$, $\mathbf{B}_{\text{cris}} = \mathbf{B}_{\text{cris}}^+[1/t] \subset \mathbf{B}_{\text{dR}}$. $\mathbf{B}_{\text{cris}}^{G_K} = K_0$. Two facts about \mathbf{A}_{cris} are: $\varphi(\xi) \in p\mathbf{A}_{\text{cris}}$, and $t^{p-1} \in p\mathbf{A}_{\text{cris}}$, hence $\mathbf{B}_{\text{cris}} = \mathbf{A}_{\text{cris}}[1/t]$. Two caveats about \mathbf{B}_{cris} are: it has a funny topology, since one can show that the topology induced on $\mathbf{B}_{\text{cris}}^+$ by \mathbf{B}_{cris} (which comes equipped with the locally convex final topology) is not the natural topology on $\mathbf{B}_{\text{cris}}^+$. Furthermore, $\mathbf{B}_{\text{cris}}^+ \subsetneq \text{Fil}^0 \mathbf{B}_{\text{cris}}$.

Let V now be a p -adic representation. Let $\mathbf{D}_{\text{dR}}(V) = (\mathbf{B}_{\text{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$. This is a K -vector space, and the injectivity of the canonical map

$$\alpha : \mathbf{B}_{\text{dR}} \otimes_K \mathbf{D}_{\text{dR}}(V) \rightarrow \mathbf{B}_{\text{dR}} \otimes_{\mathbb{Q}_p} V, \quad b \otimes (\sum b_n \otimes v_n) \mapsto \sum b b_n \otimes v_n$$

shows that $\dim_K \mathbf{D}_{\text{dR}}(V) \leq \dim_{\mathbb{Q}_p} V$. If equality holds, we call V **de Rham**. $\mathbf{D}_{\text{dR}}(V)$ comes equipped with a separated and exhaustive K -vector space filtration, given by $\text{Fil}^i \mathbf{D}_{\text{dR}}(V) = (\text{Fil}^i \mathbf{B}_{\text{dR}} \otimes V)^{G_K}$.

Similarly, we let $\mathbf{D}_{\text{cris}}(V) := (\mathbf{B}_{\text{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$. This is a K_0 -vector space, and the injectivity of the analogous α -map for \mathbf{B}_{cris} shows that $\dim_{K_0} \mathbf{D}_{\text{cris}}(V) \leq \dim_{\mathbb{Q}_p} V$. If equality holds, we call V **crystalline**. $\mathbf{D}_{\text{cris}}(V)$ comes equipped with a K_0 -linear φ -action. Further, $K \otimes_{K_0} \mathbf{D}_{\text{cris}}(V)$ comes equipped with a K -vector space filtration. If V is crystalline then V is de Rham.

One fundamental theorem of Colmez and Fontaine states: the assignment $V \mapsto \mathbf{D}_{\text{cris}}(V)$ induces an equivalence of categories between the crystalline representations of G_K and the category of K -filtered admissible (i.e. $t_H(D) = t_N(D)$ and $t_H(D') \leq t_N(D')$ for all subobjects $D' \subset D$, where t_H resp. t_N are the Hodge number resp. the Newton number) φ -modules. This equivalence heavily uses the fact that the map α above in the \mathbf{B}_{cris} -case is actually an isomorphism.

3 The period ring \mathbf{A}_{exp}

Definition 3.1. *Let*

$$\tilde{\mathbf{A}}^+ \left\{ \frac{\pi}{p} \right\} := \tilde{\mathbf{A}}^+ \{X\} / (pX - \pi),$$

where $A\{X\}$ denotes the p -adic completion of $A[X]$ for any ring A , equipped with quotient topology.

If $x \in \tilde{\mathbf{A}}^+ \{\pi/p\}$, we may write (non-uniquely) $x = \sum_{n \geq 0} a_n (\pi/p)^n$ in $\tilde{\mathbf{A}}^+[1/p]$. The natural actions of φ and G_K extend to actions on $\tilde{\mathbf{A}}^+ \{\pi/p\}$.

Lemma 3.2. *In $\tilde{\mathbf{A}}^+ \{\pi/p\} \subset \mathbf{B}_{\text{dR}}^+$ one has the relation $t/p = \pi/p \cdot v$ mit $v \in \tilde{\mathbf{A}}^+ \{\pi/p\}^\times$, i.e. $\tilde{\mathbf{A}}^+ \{\pi/p\} = \tilde{\mathbf{A}}^+ \{t/p\}$. In particular, $t \in \tilde{\mathbf{A}}^+ \{\pi/p\}$.*

Proof. First, we observe that

$$\frac{t}{p} = \sum_{n \geq 1} (-1)^{n+1} \cdot \frac{\pi^n}{p \cdot n} = \frac{\pi}{p} \cdot \left(\sum_{n \geq 1} a_n \left(\frac{\pi}{p} \right)^n \right) = \pi/p \cdot v,$$

with $a_n \rightarrow 0$ p -adically.

Now, since $v \bmod p \in \tilde{\mathbf{E}}^+[X]/(\bar{\pi})$ is -1, hence a unit, we have that $v \in \tilde{\mathbf{A}}^+ \{\pi/p\}^\times$. Hence the claim. \square

Definition 3.3. *Let A be a subring of \mathbf{B}_{dR} , such that the Frobenius φ acts on A (e.g. \mathbf{A}_{cris}). Set*

$$\text{Fil}_p^i A := \{x \in \text{Fil}^i A \mid \varphi(x) \in p^i A\},$$

where $(\text{Fil}^i A)_{i \in \mathbb{Z}}$ is the filtration induced by \mathbf{B}_{dR} .

Definition 3.4. Let $\mathbf{A}_{\exp} := \tilde{\mathbf{A}}^+\{\pi/p\}[p/t]$. The Frobenius φ on $\tilde{\mathbf{A}}^+\{\pi/p\}$ extends to \mathbf{A}_{\exp} . We equip \mathbf{A}_{\exp} with the filtration given by

$$\mathrm{Fil}^i \mathbf{A}_{\exp} := \bigcup_{i+k \geq 0} \left(\frac{p}{t}\right)^k \mathrm{Fil}_p^{i+k} \tilde{\mathbf{A}}^+ \left\{ \frac{\pi}{p} \right\}.$$

Since $\mathrm{Fil}_p^0 \tilde{\mathbf{A}}^+\{\pi/p\} = \tilde{\mathbf{A}}^+\{\pi/p\}$, this filtration is separated and exhaustive.

Proposition 3.5. For every $k \geq 0$ we have the exact G_K -equivariant sequence

$$0 \longrightarrow \left(\frac{t}{p}\right)^k \cdot \mathbb{Z}_p \longrightarrow \mathrm{Fil}_p^k \tilde{\mathbf{A}}^+\{\pi/p\} \xrightarrow{1-p^{-k}\varphi} \tilde{\mathbf{A}}^+\{\pi/p\} \longrightarrow 0,$$

which admits a continuous (not necessarily G_K -equivariant) splitting $\tilde{\mathbf{A}}^+\{\pi/p\} \rightarrow \mathrm{Fil}_p^k \tilde{\mathbf{A}}^+\{\pi/p\}$.

Proof. Obviously,

$$\left(\frac{t}{p}\right)^k \cdot \mathbb{Z}_p \subset \ker(1 - p^{-k}\varphi).$$

On the other hand, if $x \in \ker(1 - p^{-k}\varphi)$ then $x = \sum_n a_n (t/p)^n$ (see Lemma 3.2), with $\tilde{\mathbf{A}}^+ \ni a_n \rightarrow 0$ p -adically. For any $n \in \mathbb{N}$ we have $(p^{-k}\varphi)^n(x) \equiv \varphi^n(a_n)(t/p)^k \pmod{p\tilde{\mathbf{A}}^+\{\pi/p\}}$, hence $x = y(t/p)^k$, with $y \in \tilde{\mathbf{A}}^+$ and $\varphi(y) = y$, that is, $y \in \mathbb{Z}_p$ as is well-known.

We now prove that $\mathrm{Fil}_p^k \mathbf{A}_{\exp}$ is the p -adic closure of the module

$$\tilde{\mathbf{A}}^+[\xi^i \cdot (t/p)^j; i+j \geq k],$$

which we denote by N . If $i+j \geq k$ one has

$$\varphi \left(\xi^i \cdot \left(\frac{t}{p}\right)^j \right) = p^{i+j} \cdot \left(1 + \frac{\pi_0}{p}\right)^i \cdot \left(\frac{t}{p}\right)^j,$$

where π_0 is the trace from $K((t))$ to $K((t^p))$ of π . Here we recall that $\pi = \exp t - 1 \in K((t))$. Obviously $(t/p)^k \cdot \mathbb{Z}_p \subset N$, so we have to prove that for any $y \in \tilde{\mathbf{A}}^+\{\pi/p\}$ there exists an $x \in \mathrm{Fil}_p^k \tilde{\mathbf{A}}^+\{\pi/p\}$ with $(1 - p^{-k}\varphi)(x) = y$. Since N and $\tilde{\mathbf{A}}^+\{\pi/p\}$ are separated and complete with respect to the p -adic topology, it suffices to show this result mod p . If $y = \sum_{n \geq k} a_n (t/p)^n$ then $x = -y$ will do the job.

Thus it remains to show that if $y \in \tilde{\mathbf{A}}^+$ and $j \leq k$ there exists an $x \in N$ such that

$$(1 - p^{-k}\varphi)(x) - y \left(\frac{t}{p}\right)^j \in \left(p, \left(\frac{t}{p}\right)^i; i > n\right) \trianglelefteq \tilde{\mathbf{A}}^+\{\pi/p\}.$$

One checks that

$$x = y'(q')^{k-j} \left(\frac{t}{p}\right)^i,$$

where $y' \in \tilde{\mathbf{A}}^+$ is a solution of

$$\varphi(y') - q'^{k-j}y' = b$$

in $\tilde{\mathbf{A}}^+$, satisfies this property (recall that $q' = \varphi^{-1}(q)$). \square

Corollary 3.6. *Dividing out $(t/p)^{-k}$ and taking the direct limit over the sequence in Proposition 3.5 we obtain an exact sequence*

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathrm{Fil}^0 \mathbf{A}_{\exp} \xrightarrow{1-\varphi} \mathbf{A}_{\exp} \longrightarrow 0,$$

where φ is the extension of the φ on $\tilde{\mathbf{A}}^+$.

Proposition 3.7. $(\mathbf{A}_{\exp})^{G_K} = \mathcal{O}_{K_0}$

Proof. Obviously, $\mathcal{O}_{K_0} \subset (\mathbf{A}_{\exp})^{G_K}$, since $(\tilde{\mathbf{A}}^+)^{G_K} = \mathcal{O}_{K_0}$. We have the inclusions

$$\varphi\left(\tilde{\mathbf{A}}^+ \left\{ \frac{\pi}{p} \right\}\right) = \tilde{\mathbf{A}}^+ \left\{ \frac{\pi^p}{p} \right\} \subset \mathbf{A}_{\mathrm{cris}} \subset \tilde{\mathbf{A}}^+ \left\{ \frac{\pi}{p} \right\},$$

which leads to

$$\varphi(\mathbf{A}_{\exp}[1/p]) \subset \mathbf{B}_{\mathrm{cris}} \subset \mathbf{A}_{\exp}[1/p].$$

Since $(\varphi(\mathbf{B}_{\mathrm{cris}}))^{G_K} = (\mathbf{B}_{\mathrm{cris}})^{G_K} = K_0$, we have that $(\varphi(\mathbf{A}_{\exp}[1/p]))^{G_K} = K_0$. Since φ is injective on \mathbf{A}_{\exp} , and hence on $\mathbf{A}_{\exp}[1/p]$, we obtain $(\mathbf{A}_{\exp}[1/p])^{G_K} = K_0$.

Note that by the above exact sequence, $1/p \notin \mathbf{A}_{\exp}$: otherwise $1/p^n \in \mathbf{A}_{\exp}$ for all $n \geq 0$. But taking $G_{\mathbb{Q}_p}$ -invariants gives an injection

$$\mathbf{A}_{\exp} \hookrightarrow H^1(G_{\mathbb{Q}_p}, \mathbb{Z}_p) = \mathrm{Hom}_{\mathrm{cts}}(G_{\mathbb{Q}_p}, \mathbb{Z}_p).$$

The \mathbb{Z}_p -module on the right hand side is finitely generated, which would lead to a contradiction.

Alternatively, one can use the exact sequence of 3.5, the filtration on \mathbf{A}_{\exp} and a limit argument to proceed as in the proof of the statement

$$H^0(K, \mathrm{Fil}^i \mathbf{B}_{\mathrm{dR}} / \mathrm{Fil}^j \mathbf{B}_{\mathrm{dR}}) = K,$$

if $i \leq 0 < j$. \square

Definition 3.8. *Let T be a full \mathbb{Z}_p -lattice of V that is invariant under the action of G_K (such lattices always exist). We define the modules*

$$\mathbf{D}_{\exp}(T) := (\mathbf{A}_{\exp} \otimes_{\mathbb{Z}_p} T)^{G_K}$$

and

$$\mathbf{D}_{\exp}^0(T) := (\mathrm{Fil}^0 \mathbf{A}_{\exp} \otimes_{\mathbb{Z}_p} T)^{G_K}.$$

Proposition 3.9. *If T is as before, $\mathbf{D}_{\exp}(T)$ is free \mathcal{O}_{K_0} -module of finite rank less or equal to $\mathrm{rk}_{\mathbb{Z}_p} T$.*

Proof. This is a variation of the proof one usually encounters in Fontaine's theory of B -admissible rings. We outline the idea:

Let B be a topological integral domain, equipped with a continuous action of a topological group G . Set $C = \text{Frac}(B)$ and $S = B^G$, which is again an integral domain, and fix a closed subring $R \subset S$. Assume T is a finite free R -module with continuous G -action, so that $V = \text{Frac}(R) \otimes_R T$ is a finite-dimensional $\text{Frac}(R)$ -vector space. Set $D_B(T) = (B \otimes_R T)^G$, $D_C(V) = (C \otimes_{\text{Frac}(R)} V)^G$ and assume $C^G = \text{Frac}(S)$. We want to prove the injectivity of the map

$$B \otimes_S D_B(T) \longrightarrow B \otimes_R T.$$

The inclusion $B \hookrightarrow C$ and the freeness of T gives a diagram

$$\begin{array}{ccc} B \otimes_S D_B(T) & \longrightarrow & B \otimes_R T \\ \downarrow & & \downarrow \\ B \otimes_S D_C(V) & & \\ \downarrow & & \downarrow \\ C \otimes_{\text{Frac}(S)} D_C(V) & \longrightarrow & C \otimes_{\text{Frac}(R)} V \end{array} \quad ,$$

so that we are reduced to the case where all the rings are fields. Now one proceeds exactly as in [3], Theorem 2.13.

The above situation applies with $B = \mathbf{A}_{\text{exp}}$, $R = \mathbb{Z}_p \subset \mathcal{O}_{K_0} = S$. The injectivity of the above map implies, by using the above notation and going to the quotient field C , that $D_B(T)$ is of S -rank smaller or equal than the R -rank of T . Since we these latter rings are discrete valuation rings, we are done. \square

Proposition 3.10. *If V and T are as above, we have*

$$\mathbf{D}_{\text{exp}}(T)[1/p] = \mathbf{D}_{\text{exp}}(V) = (\mathbf{A}_{\text{exp}}[1/p] \otimes_{\mathbb{Q}_p} V)^{G_K} = \mathbf{D}_{\text{cris}}(V),$$

and this identification is compatible with the K -filtrations and the action of the Frobenius φ .

Proof. The proof is similarly as in Proposition 3.7. We have inclusions

$$(\varphi(\mathbf{A}_{\text{exp}}) \otimes T)^{G_K} \subset (\mathbf{A}_{\text{cris}} \otimes T)^{G_K} \subset (\mathbf{A}_{\text{exp}} \otimes T)^{G_K},$$

and since φ is bijective on $\mathbf{D}_{\text{cris}}(V)$ and injective on \mathbf{A}_{exp} , we conclude as before $(\mathbf{A}_{\text{exp}}[1/t] \otimes T)^{G_K} = \mathbf{D}_{\text{cris}}(V)$.

The compatibility with filtration and Frobenius can be checked by the construction. \square

4 Categories in integral p -adic Hodge theory

Let K/\mathbb{Q}_p be unramified for this section.

Definition 4.1. A *Fontaine-Laffaille module* over \mathcal{O}_K is a triple $(M, (M^i)_{i \in \mathbb{Z}}, (\varphi^i)_{i \in \mathbb{Z}})$, which we also denote simply by M , consisting of

- an \mathcal{O}_K -module M ,
- an exhausting and separated decreasing filtration (of \mathcal{O}_K -modules) M^i of M ,
- a family of σ -semilinear maps $\varphi^i : M^i \rightarrow M$ with the property $\varphi^i|_{M^{i+1}} = p \cdot \varphi^{i+1}$.

A morphism of Fontaine-Laffaille modules $f : M \rightarrow N$ is a \mathcal{O}_K -linear map f such that $f(M^i) \subset N^i$ and $f \circ \varphi_M^i = \varphi_N^i \circ f$. We denote by $\underline{MF}_{\mathcal{O}_K}$ the exact category of all Fontaine-Laffaille modules over \mathcal{O}_K .

Definition 4.2. A *filtered Dieudonné module* over \mathcal{O}_K is a Fontaine-Laffaille module M such that

- M is of finite type over \mathcal{O}_K ,
- $M^i = M$ for $i \ll 0$ and $M^j = 0$ for $j \gg 0$,
- $M_\sigma = \sum_{i \in \mathbb{Z}} \varphi^i(M^i)$.

We denote by $\underline{MF}_{\mathcal{O}_K}^{\text{fd}}$ (fortement divisible) the category of all filtered Dieudonné modules.

Here, M_σ denotes the underlying module M , where W acts via σ . Note that contrary to the usual convention we allow our Dieudonné modules to contain torsion.

Example 4.3. We have $\mathbf{A}_{\text{exp}} = (\mathbf{A}_{\text{exp}}, (F^i \mathbf{A}_{\text{exp}})_{i \in \mathbb{Z}}, (\varphi^i)_{i \in \mathbb{Z}}) \in \underline{MF}$, where \mathbf{A}_{exp} and the filtration are given as before, and $\varphi^i = 1/p^i \cdot \varphi$, with φ induced from the Frobenius on \mathbf{A}^+ .

Theorem 4.4. The category $\underline{MF}_{\mathcal{O}_K}^{\text{fd}}$ is abelian.

Proof. This follows from the fact that $\underline{MF}_{\mathcal{O}_K}^{\text{fd,tor}}$, the subcategory of all torsion \mathcal{O}_K -modules ([2], Proposition 1.8), is abelian, and completeness: let $f : M \rightarrow N$ be a map in $\underline{MF}_{\mathcal{O}_K}$ with $M, N \in \underline{MF}_{\mathcal{O}_K}^{\text{fd}}$. This gives us, for any $n \in \mathbb{N}$, a map $f_n : M/p^n \rightarrow N/p^n$ in $\underline{MF}_{\mathcal{O}_K}^{\text{fd,tor}}$, since M and N are of finite type, hence kernel and cokernel of f_n exist.

Since $M = \varprojlim M/p^n$, in a compatible way with the filtration and the Frobenii, we obtain, by going to the limit, the kernel and the cokernel of the map f . The normality of mono- and epimorphisms is an easy consequence again of the property that $\underline{MF}_{\mathcal{O}_K}^{\text{fd,tor}}$ is abelian. \square

Definition 4.5. A *filtered φ -module* over K is a triple $(D, (D^i)_{i \in \mathbb{Z}}, \varphi)$, consisting of

- a K -vector space D ,
- an exhausting and separated decreasing filtration (of K -vector spaces) D^i of D ,
- a σ -semilinear map $\varphi : D \rightarrow D$.

A morphism of filtered φ -modules is, similarly as before, a morphism of K -vector spaces compatible with the filtration and Frobenius φ . We denote by \underline{MF}_K the category of all φ -modules.

A finite-dimensional filtered φ -module is called **admissible** if $t_N(D) = t_H(D)$ and $t_N(D') \leq t_H(D')$ for all subobjects $D' \subset D$ in \underline{MF}_K , where t_N resp. t_H are the Newton- resp. Hodge number of D (see [3], 6.4.2.).

Example 4.6. If $M \in \underline{MF}_{\mathcal{O}_K}^{\text{fd}}$, one can naturally associate a finite-dimensional φ -module D to M , namely $D := K \otimes_{\mathcal{O}_K} M$, with the filtration induced by M^i , and Frobenius $\varphi := 1/p^n \cdot \varphi^n$ for $n \ll 0$. We call M **admissible** if D is admissible.

Proposition 4.7. Let $D \in \underline{MF}_K$ be admissible. Then an \mathcal{O}_K -lattice M , equipped with a filtration M^i such that $M^i[1/p] = D^i$ can be considered as an object of $\underline{MF}_{\mathcal{O}_K}^{\text{fd}}$ if and only if $\varphi(M^i) \subset p^i M$ (that is, one puts $\varphi^i := p^{-i}\varphi$).

Proof. The only thing we have to check is that the condition on φ holds, then D is already in $\underline{MF}_{\mathcal{O}_K}^{\text{fd}}$. This can be inferred by the proof of [2], Theorem 3.2, after reducing to the case where all weights are ≥ 0 (see also [4], Proposition 7.8). \square

Proposition 4.8. If V is crystalline and T as above, then $\mathbf{D}_{\text{exp}}(T) \in \underline{MF}_{\mathcal{O}_K}^{\text{fd}}$.

Proof. We know from proposition 3.9 that $\mathbf{D}_{\text{exp}}(T)$ is a free \mathcal{O}_K -module of finite rank and that $\mathbf{D}_{\text{exp}}(T)[1/p] = \mathbf{D}_{\text{cris}}(V)$ (3.10) is admissible, since V is crystalline. Since $\mathbf{D}_{\text{exp}}^i(T)[1/p] = \text{Fil}^i \mathbf{D}_{\text{cris}}(V)$ and $\varphi^i(\mathbf{D}_{\text{exp}}^i(T)) \subset \mathbf{D}_{\text{exp}}(T)$, the requirements of proposition 4.7 are fulfilled, hence the claim. \square

5 Computation of $H_e^1(K, T)$

Assume again that K/\mathbb{Q}_p is unramified. We collect some facts from sections 3 and 4 of [1].

Proposition 5.1. Let $M \in \underline{MF}_{\mathcal{O}_K}^{\text{fd}}$ and put

$$H^0(M) = \ker((1 - \varphi^0) : M^0 \rightarrow M), \quad H^1(M) = \text{coker}((1 - \varphi^0) : M^0 \rightarrow M).$$

and $H^i(M) = 0$ for $i \geq 2$. Then $(H^i)_{i \in \mathbb{N}}$ is a cohomological δ -functor.

Proof. This is abundantly clear by the snake lemma. \square

Recall ([1], Proposition 1.17) the Bloch-Kato fundamental exact sequences

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow \mathbf{B}_{\text{cris}}^{\varphi=1} \oplus \mathbf{B}_{\text{dR}}^+ \xrightarrow{f} \mathbf{B}_{\text{dR}} \longrightarrow 0,$$

where $f(x, y) = x - y$, and

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow \mathbf{B}_{\text{cris}} \oplus \mathbf{B}_{\text{dR}}^+ \xrightarrow{f} \mathbf{B}_{\text{cris}} \oplus \mathbf{B}_{\text{dR}} \longrightarrow 0,$$

where $f(x, y) = ((1 - \varphi)(x), x - y)$. If V is a p -adic representation, we call the maps

$$\exp_e : \mathbf{D}_{\text{dR}}(V) \xrightarrow{\delta} H^1(K, V), \quad \exp_f : \mathbf{D}_{\text{cris}}(V) \oplus \mathbf{D}_{\text{dR}}(V) \xrightarrow{\delta} H^1(K, V)$$

the **Bloch-Kato exponential maps**, which are induced by the connecting homomorphism of Galois cohomology. We set

$$H_e^1(K, V) := \text{Im}(\exp_e), \quad H_f^1(K, V) := \text{Im}(\exp_f).$$

If $T \subset V$ is a G_K -equivariant \mathbb{Z}_p -lattice in V , denote by $\iota : H^1(K, T) \rightarrow H^1(K, V)$ the canonical map. Set

$$H_f^1(K, T) := \text{Im}(\exp_f)$$

Recall that the local L -function in the case $p = l$ is defined as

$$P(V, X) := \det_K(1 - f \cdot X \mid \mathbf{D}_{\text{cris}}(V)),$$

where f denotes the K -linear map $\varphi^{[K:\mathbb{Q}_p]}$.

Assume now that $P(V, 1) \neq 0$. Then (cf. the first lines in the proof of Theorem 4.1, [1]) $H_f^1(K, V) = H_e^1(K, V)$.

Theorem 5.2. *Let V be a crystalline representation, fix a G_K -equivariant lattice T in V , and assume $P(V, 1) \neq 0$. Then:*

- a) $H^1(\mathbf{D}_{\text{exp}}(T))[1/p] \cong H_e^1(K, V)$.
- b) $H^1(\mathbf{D}_{\text{exp}}(T)) \cong H_e^1(K, T)$.
- c) $\exp_e : \mathbf{D}_{\text{dR}}(V)/\mathbf{D}_{\text{dR}}^0(V) \rightarrow H^1(K, V)$ coincides with the composite map

$$\begin{aligned} (\mathbf{D}_{\text{exp}}(T)/\mathbf{D}_{\text{exp}}^0(T))[1/p] &\xrightarrow{1-\varphi} (\mathbf{D}_{\text{exp}}(T)/(1-\varphi^0)\mathbf{D}_{\text{exp}}(T))[1/p] \\ &= H^1(\mathbf{D}_{\text{exp}}(T))[1/p] \cong H^1(K, V), \end{aligned}$$

where the last canonical identification is explained in the proof.

Proof. The proof is similar to [1], Lemma 4.5. We have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}_p & \longrightarrow & \text{Fil}^0 \mathbf{A}_{\text{exp}} & \longrightarrow & \mathbf{A}_{\text{exp}} \longrightarrow 0 \\ & & \downarrow & & \downarrow x \mapsto (x, x) & & \downarrow x \mapsto (x, 0) \\ 0 & \longrightarrow & \mathbb{Q}_p & \longrightarrow & \mathbf{A}_{\text{exp}}[1/p] \oplus \mathbf{B}_{\text{dR}}^+ & \longrightarrow & \mathbf{A}_{\text{exp}}[1/p] \oplus \mathbf{B}_{\text{dR}} \longrightarrow 0 \end{array}$$

Inverting p , tensoring with V over \mathbb{Q}_p and taking G_K -cohomology, we obtain a diagram

$$\begin{array}{ccccccc}
0 \twoheadrightarrow H^0(\mathbf{D}_{\exp}(T))[1/p] & \longrightarrow & \mathbf{D}_{\exp}^0(T)[1/p] & \longrightarrow & \mathbf{D}_{\exp}(T)[1/p] & \longrightarrow & H^1(\mathbf{D}_{\exp}(T))[1/p] \twoheadrightarrow 0 \\
\downarrow = & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow H^0(K, V) & \longrightarrow & \mathbf{D}_{\text{cris}}(V) \oplus \mathbf{D}_{\text{dR}}^+(V) & \twoheadrightarrow & \mathbf{D}_{\text{cris}}(V) \oplus \mathbf{D}_{\text{dR}}(V) & \longrightarrow & H^1(K, V)
\end{array}$$

with exact rows. Since $\mathbf{D}_{\exp}(T)[1/p] = \mathbf{D}_{\text{cris}}(V) = \mathbf{D}_{\text{dR}}(V)$ and $\text{Im}(\exp_f) = \text{Im}(\exp_e)$, we have the claimed identification $H^1(\mathbf{D}_{\exp}(T))[1/p] \cong H_f^1(K, V)$.

The exact sequence

$$0 \longrightarrow \mathbf{D}_{\exp}(T) \xrightarrow{p} \mathbf{D}_{\exp}(T) \longrightarrow \mathbf{D}_{\exp}(T)/p \longrightarrow 0$$

(which exists in $\underline{\text{MF}}_{\mathcal{O}_K}^{\text{fd}}$, since $\mathbf{D}_{\exp}(T) \in \underline{\text{MF}}_{\mathcal{O}_K}^{\text{fd}}$) induces a sequence

$$H^0(\mathbf{D}_{\exp}(T)/p) \longrightarrow H^1(A \otimes T) \longrightarrow H^1(A \otimes T) \longrightarrow H^1(A \otimes T/p).$$

Recall that $H_f^1(K, T) = \iota^{-1}(H_f^1(K, V))$, where $\iota : H^1(K, T) \rightarrow H^1(K, V)$. Since $H^1(\mathbf{D}_{\exp}(T)) \subset H^1(K, T)$ it suffices to show that the cokernel of this inclusion does not have any p -torsion. But this follows from the commutative diagram

$$\begin{array}{ccccccc}
H^0(\mathbf{D}_{\exp}(T)/p) & \longrightarrow & H^1(A \otimes T) & \xrightarrow{p} & H^1(A \otimes T) & \longrightarrow & H^1(A \otimes T/p) \\
\downarrow \cong & & \downarrow \subset & & \downarrow \subset & & \downarrow \subset \\
H^0(T/p) & \longrightarrow & H^1(T) & \xrightarrow{p} & H^1(T) & \longrightarrow & H^1(T/p)
\end{array}$$

□

Corollary 5.3. *Let μ be the Haar-measure on the finite-dimensional K -vector space $H^1(K, V)$ such that the image of the lattice*

$$\mathbf{D}_{\exp}(T)/\mathbf{D}_{\exp}^0(T) \subset \mathbf{D}_{\text{dR}}(V)/\mathbf{D}_{\text{dR}}^0(V) \xrightarrow{\sim} H_e^1(K, V) = H_f^1(K, V)$$

has measure 1. Then

$$\mu(H_f^1(K, T)) = |P(V, 1)|_p^{-1}.$$

Proof. This follows from the definition of $P(V, X)$ and the b), c) from the previous theorem. □

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